On definitions of relatively hyperbolic groups

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Abstract. The purpose of this note is to provide a short alternate proof of the fact that [**9**, Question 1] has an affirmative answer. Our proof combined with the result of Szczepanski [**9**] shows that a group which is relatively hyperbolic in the sense of the definition of Gromov is relatively hyperbolic in the sense of the definition of Farb.

1. Definitions

The notion of a relatively hyperbolic group was introduced by Gromov [**7**] as a generalization of the concept of a (word) hyperbolic group.

1.1. The Gromov definition.

Let X be a hyperbolic (in the Gromov sense) complete locally compact geodesic metric space. Let $x \in X$, suppose z is a point at infinity, and γ is a geodesic ray from x to z. By a *horosphere through x with center z* we mean the limit as $t \to \infty$ of the sphere of radius t in X with center $\gamma(t)$. A horosphere is the level surface of the horofunction $h(x)$ corresponding to the ray γ. By the *radius* of a horosphere through x we mean the value $h(x)$. A *horoball* is the interior of a horosphere.

Suppose a group G admits a properly discontinuous isometric action on X so that the quotient space $Y = X/G$ is quasi-isometric to the union of k copies of $[0, \infty)$ joined at zero. Assume that the action of G on X is free. Lift the rays in Y to the rays $\gamma_i \colon [0, \infty) \to X$ for $i = 1, 2, \ldots, k$. Let H_i be the isotropy subgroup of $\gamma_i(\infty)$; assume that H_i preserves h_i. Assume that in X there exists a G-invariant system GB of disjoint horoballs, and the action of G on $X \setminus GB$ is cocompact.

Then Gromov calls G hyperbolic relative to the subgroups $H_1, \ldots, H_k$.

1.2. Definitions proposed by Farb.

Definition 1.1. [**6**](**Relatively hyperbolic group in a weak sense**) Let G be a finitely generated group, and let H be a finitely generated subgroup of G. Fix a set A of generators of G. In the Cayley graph $\Gamma(G, A)$ add a vertex $v(gH)$ for each left coset gH of H, and connect $v(gH)$ with each $x \in gH$ by an edge of length $\frac{1}{2}$. The obtained graph $\hat{\Gamma}$ is called a coned-off graph of G with respect to H.

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The group G is called *weakly hyperbolic relative to* H if $\hat{\Gamma}$ is a hyperbolic metric space.

Remark. The terminology in Definition 1.1 was part of what was suggested by Bowditch in [**1**]. In [**6**] and [**9**] a group G that satisfies Definition 1.1 is termed simply "hyperbolic relative to H".

THEOREM 1.2. [**9**, Theorem 1] *Let G be a finitely generated group, and let $H_1, \dots, H_r$ be a finite set of finitely generated subgroups of G. If G is hyperbolic relative to $H_1, \dots, H_r$ in the sense of Gromov's definition, then G is weakly hyperbolic relative to $H_1, \dots, H_r$.*

Furthermore, in [**9**, Example 3] Szczepanski shows that the class of weakly relatively hyperbolic groups is strictly larger than the class of groups relatively hyperbolic in the sense of Gromov's definition.

In [**6**] Farb shows solvability of the word problem for weakly relatively hyperbolic groups that have a property which he calls the Bounded Coset Penetration property. The Bounded Coset Penetration property appears to be crucial for solvability of the conjugacy problem for relatively hyperbolic groups [**2**].

DEFINITION 1.3. [**6**](**Bounded Coset Penetration property**) Let a group G be weakly hyperbolic relative to a finitely generated subgroup H. A path u in Γ is a *relative P-quasigeodesic*, if its projection $\hat{u}$ to $\hat{\Gamma}$ is a P-quasigeodesic. The path u is a *path without backtracking* if u never returns to a subset which u penetrates. The pair (G, H) is said to satisfy the *Bounded Coset Penetration (BCP)* property if $\forall P \geq 1$, there is a constant $c = c(P)$ so that for every pair u, v of relative P-quasigeodesics without backtracking, with same endpoints, the following conditions hold:

(1) If u penetrates a coset gH and v does not penetrate gH, then u travels a Γ-distance of at most c in gH.
(2) If both u and v penetrate a coset gH, then the vertices in Γ at which u and v first enter (last exit) gH lie a Γ-distance of at most c from each other.

Thus one considers the class of groups which can be defined as follows.

DEFINITION 1.4. (**Relatively hyperbolic group by Farb**) Let G be a finitely generated group, and let H be a finitely generated subgroup of G. We say that G is *hyperbolic relative to H in the sense of Farb*, if G is weakly hyperbolic relative to H and the pair (G, H) has the BCP property.

In [**9**] Szczepanski asks the following question.

QUESTION 1.5. [**9**, Question (1)] Let a group G be hyperbolic relative to a finitely generated subgroup H in the sense of Gromov's definition. Does the pair (G, H) have the BCP property?

It is already known that Question 1.5 has an affirmative answer. In fact, results of Bowditch [**1**] and Dahmani [**4**],[**5**] imply that the class of groups that are relatively hyperbolic in the sense of Gromov's definition and the class of groups that are relatively hyperbolic in the sense of Farb's definition coincide. We provide a direct proof of the affirmative answer to Question 1.5. Combined with the proof of Theorem 1.2 by Szczepanski, this gives a short alternate proof of the following theorem.

THEOREM 1.6. *Let G be a finitely generated group, and let H be a finitely generated subgroup of G. If the group G is hyperbolic relative to H in the sense of the Gromov definition, then G is hyperbolic relative to H in the sense of the definition of Farb.*

2. Quasiconvex subsets in a hyperbolic space

Our approach is to generalize the arguments that Farb used to prove [**6**, Theorem 4.11] (cf. [**8**, Theorem 5.10]).

NOTATION 2.1. Let (X, d) be a δ-hyperbolic metric space ($\delta \geq 0$) with a collection Σ of closed disjoint ϵ-quasiconvex subsets. We assume that the distance between any two subsets $S_1, S_2 \in \Sigma$ is bounded below by a constant $R > 24\delta + 4\epsilon$. We obtain a space X_S by deleting the interiors of all of the sets in Σ. The boundary of X_S consists of disjoint connected components, each component is the boundary of a set $S \in \Sigma$. We give X_S the path metric d_S. Next, we obtain the quotient $\hat{X}$ of X_S by identifying points which lie in the same boundary component of X_S. Hence $\hat{X}$ is equipped with a path pseudometric $\hat{d}$ induced from the path metric d_S.

LEMMA 2.2. [**9**, Proposition 1]. *The space $(\hat{X}, \hat{d})$ is hyperbolic in the Gromov sense.*

DEFINITION 2.3. (**Projections onto quasiconvex sets**) Fix a subset $S \in \Sigma$. Let x and y be two points in X, and let $Pr(x)$ and $Pr(y)$ denote the projections of these points onto S. Let α be a path in X with the initial point $i(\alpha) = x$ and the terminal point $t(\alpha) = y$. By the *projection $Pr(\alpha, S)$ of α onto S* we mean an X-geodesic connecting $Pr(x)$ and $Pr(y)$, and by the *X-length $l_X(Pr(\alpha, S))$ of the projection $Pr(\alpha, S)$* we mean the X-length of that geodesic:

$$l_X(Pr(\alpha, S)) = d(Pr(x), Pr(y)).$$

Given $S_1 \in \Sigma$, by the *length of the projection of S_1 onto S* we mean the maximum length of the projection of a path in S_1 onto S:

$$l_X(Pr(S_1, S)) = \max\{l_X(Pr(\alpha, S)) \mid \alpha \subset S_1\}.$$

NOTATION 2.4. Let γ be a path in X, and let $l_X(\gamma)$ be the X-length of γ. We denote by $\hat{\gamma}$ the projection of γ into $\hat{X}$, and by $l_{\hat{X}}(\hat{\gamma})$ the $\hat{X}$-length of $\hat{\gamma}$. Obviously, $l_{\hat{X}}(\hat{\gamma}) \leq l_X(\gamma)$. Also, given a path $\hat{\gamma}$ in $\hat{X}$, by γ we mean a path in X whose projection into $\hat{X}$ is $\hat{\gamma}$. Given $\hat{\gamma}$, the path γ is not unique in general, but the following definition does not depend on particular choice of γ.

DEFINITION 2.5. (**Intersections with quasiconvex sets**) We will say that $\hat{\gamma}$ (or γ) *intersects* a subset $S \in \Sigma$, if γ intersects the boundary of S and travels a non-zero X-distance in the interior of S.

By [**7**, Lemma 7.3D] and [**3**, Chapter 10, Proposition 2.1], we have the inequality

$$d(Pr(x), Pr(y)) \leq \max(C, C + d(x, y) - d(x, Pr(x)) - d(y, Pr(y))), \tag{1}$$

where $C = 2\epsilon + 12\delta$. Moreover, in [**9**] Szczepanski shows the following. If the geodesic segment $Pr(\alpha, S)$ does not intersect the 2δ-neighborhood of α in X, then

$$l_X(Pr(\alpha, S)) \leq C. \tag{2}$$

Assume that each $S \in \Sigma$ is a convex set. In the proof of Theorem 1.6 (Section 3 below) we use the fact that horoballs are convex sets. Let ∂S denote the boundary of a set $S \in \Sigma$. Observe that ∂S may be not convex (for instance, horospheres are not convex). For a set $U \subset X$, denote by $Nb_X(U, \lambda)$ the λ-neighborhood of U in X.

LEMMA 2.6. *Let $S \in \Sigma$ be a set, and let β be an X-geodesic that does not intersect $Nb_X(S, 2\delta)$. If x and y are the endpoints of β, then*

$$d_S(Pr(x), Pr(y)) \leq C + 16\delta.$$

PROOF. Let $x_s = Pr(x, S)$ and $y_s = Pr(y, S)$, so that $[x, x_s] \cap S = x_s$ and $[y, y_s] \cap S = y_s$. By the inequality (2), $d(x_s, y_s) \leq C$. Moreover, X-geodesic $[x_s, y_s]$ stays 4δ-close to the union of the geodesics $[x_s, x] \cup \beta \cup [y, y_s]$. Hence, there exists a path in $X \setminus S$ which joins x_s and y_s, with X-length bounded by $C + 16\delta$, as claimed. □

LEMMA 2.7. *Let $\hat{\gamma}$ be a P-quasigeodesic ($P \geq 1$) in $\hat{X}$. Assume that $\hat{\gamma}$ does not intersect any subset $S \in \Sigma$. Given a subset $S_0 \in \Sigma$, the projection $Pr(\gamma, S_0)$ of γ onto S_0 has d_S-length at most $D = D(P)$.*

PROOF. Let x and y denote the endpoints of $\hat{\gamma}$, and let β be an X-geodesic joining x and y. Observe that $\gamma = \hat{\gamma}$, so that $l_X(\gamma) = l_{\hat{X}}(\hat{\gamma})$. Since $\hat{d}(x, y) \leq d(x, y)$, γ is a P-quasigeodesic in X. If β does not intersect $Nb_X(S, 2\delta)$, then the claim follows from Lemma 2.6. In what follows, we assume that β intersects $Nb_X(S, 2\delta)$. Let z (or w) be the point where β first enters (or last exits) $Nb_X(S, 2\delta)$, and let $z_s = Pr(z, S_0)$ and $w_s = Pr(w, S_0)$. It remains to show that $d_S(z_s, w_s)$ is bounded. We argue as follows.

The Hausdorff X-distance between β and γ is bounded by a constant $N(P)$, hence there are points $a, b \in \gamma$ so that $d(a, z) \leq N(P)$ and $d(b, w) \leq N(P)$. Denote by γ_0 the subsegment of γ between a and b. Having projected γ to $\hat{X}$, we have the following inequalities:

$$\hat{d}(a, b) \leq d(a, Pr(a, S)) + d(b, Pr(b, S)) \leq 2(N(P) + 2\delta), \quad \text{so that}$$
$$l_{\hat{X}}(\hat{\gamma_0}) \leq 2P(N(P) + 2\delta).$$

As $l_X(\gamma_0) = l_{\hat{X}}(\hat{\gamma_0})$ and $d_S(z_s, w_s) \leq d(z_s, a) + l_X(\gamma_0) + d(b, w_s)$, we have that

$$d_S(z_s, w_s) \leq 2(N(P) + 2\delta) + 2P(N(P) + 2\delta). \tag{3}$$

Since Lemma 2.6 applies to the segments $[x, z]$ and $[w, y]$, we conclude that the d_S-length of the projection $Pr(\gamma, S_0)$ of γ onto S_0 is bounded by

$$D(P) = 2(C + 16\delta) + 2(P + 1)(N(P) + 2\delta)). \tag{4}$$

□

For each P-quasi-geodesic $\hat{\gamma}$ in $\hat{X}$, we are able to bound the d_S-length of the projection of $\hat{\gamma}$ onto a quasiconvex set in terms of the length $l_{\hat{X}}(\hat{\gamma})$.

LEMMA 2.8. *Fix a subset $S_0 \in \Sigma$. Let $P \geq 1$ and $\hat{\gamma}$ be a P-quasigeodesic in $\hat{X}$, which does not intersect S_0. Then the projection $Pr(\gamma, S_0)$ of γ onto S_0 has d_S-length at most*

$$l_S(Pr(\gamma), S_0) \leq E(\hat{\gamma}, P) = \big(\frac{2l_{\hat{X}}(\hat{\gamma})}{R} + 3\big)(C + 16\delta + D(P)). \tag{5}$$

PROOF. Let $S \in \Sigma$ be a subset different from S_0. Since S and S_0 stay distance at least R apart, by Lemma 2.6, the projection of S onto S_0 has d_S-length bounded by $C + 16\delta$. Moreover, $\hat{\gamma}$ intersects at most $n = \frac{l_{\hat{X}}(\hat{\gamma})}{R} + 1$ quasiconvex subsets from Σ, and has at most $n + 2$ subsegments which do not intersect any $S \in \Sigma$. By Lemma 2.7, the claim follows. $\square$

LEMMA 2.9. [**9**, Lemma 1] *For any $K \geq 2\delta+\epsilon$, there is a constant $L = L(R, K)$ so that in the $\hat{X}$-metric, whenever $\hat{\alpha}$ is an $\hat{X}$-geodesic, $\hat{\alpha}$ stays $(K + L/2)$-close to an X-geodesic β with the same endpoints as $\hat{\alpha}$.*

As a consequence of Lemma 2.8 and Lemma 2.9, we have a uniform bound on the projection length of an $\hat{X}$-geodesic, which does not depend on the length of this geodesic.

LEMMA 2.10. *Let $K \geq 2\delta+\epsilon$, and let $\hat{\alpha}$ be a $\hat{X}$-geodesic so that α does not intersect the K-neighborhood $Nb_X(S_0, K)$ of S_0 in X. Then the d_S-length of $Pr(\alpha, S_0)$ is bounded by a constant $E_1 = E_1(R, K, \delta)$ which does not depend on the length of $\hat{\alpha}$.*

PROOF. Whenever $a \in X$ is a point, by a_s we mean the projection of a onto S_0. Let x and y be the endpoints of $\hat{\alpha}$, and let β be an X-geodesic joining x and y. Observe that if β does not intersect $Nb_X(S_0, 2\delta)$, then by Lemma 2.6, $d_S(x_s, y_s) \leq C + 16\delta$. In what follows, we assume that β intersects $Nb_X(S_0, 2\delta)$. Let z (or w) be the point where β first enters (or last exits) $Nb_X(S_0, 2\delta)$. By Lemma 2.9, $\hat{\alpha}$ stays $(K + L/2)$-close to β in the $\hat{X}$-metric. Therefore, there are points $a, b \in \hat{\alpha}$ so that $\hat{d}(a, z) \leq K + L/2$ and $\hat{d}(b, w) \leq K + L/2$. Hence,

$$l_{\hat{X}}(\hat{\alpha}_0) \leq 2(K + L/2 + 2\delta), \tag{6}$$

where $\hat{\alpha}_0$ is the segment of $\hat{\alpha}$ joining a and b. By Lemma 2.8, we have that

$$d_S(a_s, b_s) \leq E(\hat{\alpha}_0) \leq E(2(K + L/2 + 2\delta), 1). \tag{7}$$

Let $\hat{\alpha}_1$ (or $\hat{\alpha}_2$) be the segment of $\hat{\alpha}$ joining x and a (or b and w). It remains to show that the projections of $\hat{\alpha}_1$ and of $\hat{\alpha}_2$ have bounded d_S-length. Let β_i be an X-geodesic joining the endpoints of $\hat{\alpha}_i$. As long as β_1 intersects $Nb_X(S_0, 2\delta)$, we are able to find a point a_1 in $\hat{\alpha}_1$ which is $(K + L/2 + 2\delta)$-close to S_0, so that the $\hat{X}$-length of the segment of $\hat{\alpha}$ joining a_1 and b satisfies the inequality (6) above. Therefore, without loss of generality, we can assume that neither β_1 nor β_2 intersects $Nb_X(S_0, 2\delta)$. Hence Lemma 2.6 applies to these segments, so that the length of the projection of $\hat{\alpha}$ is bounded as follows:

$$l_S(Pr(\alpha), S_0) \leq E_1 = E(2K + L + 4\delta, 1) + 2(C + 16\delta). \tag{8}$$

This upper bound does not depend on the length of $\hat{\alpha}$, as claimed. $\square$

LEMMA 2.11. ***(Bounded Subset Penetration)*** *Let $P \geq 1$ be a constant, and let $\hat{\xi}$ and $\hat{\tau}$ be two P-quasigeodesics without backtracking in $\hat{X}$, with common endpoints. Then there exists a constant $B = B(\delta, \epsilon, P)$ such that the following holds.*

(1) *If $\hat{\xi}$ intersects a subset $S \in \Sigma$ and $\hat{\tau}$ does not intersect S, then the d_S-distance s between the points where ξ first enters and last exits S is bounded by B.*

(2) *If both $\hat{\xi}$ and $\hat{\tau}$ intersect a subset $S \in \Sigma$, then the d_S-distance s between the points where $\hat{\xi}$ and $\hat{\tau}$ first enter (or last exit) S is bounded by B.*

Proof. Fix $K = 2\delta + \epsilon$. In the case (1), observe that s is bounded by the d_S-length of the projection of the path which is the concatenation of $\hat{\tau}$ and the two subsegments of $\hat{\xi}$ that lie outside S. Let $\hat{\alpha}$ denote either of these P-quasigeodesic segments, and let $\hat{\beta}$ be a $\hat{X}$-geodesic with the same endpoints as $\hat{\alpha}$. Observe that if a subsegment $\hat{\beta}'$ of $\hat{\beta}$ does not intersect $Nb_{\hat{X}}(S, K)$, then β' does not intersect $Nb_X(S, K)$, hence by Lemma 2.10, we have that $l_S(Pr(\beta')) \leq E_1$.

Next , assume that $\hat{\beta}$ intersects $Nb_X(S, K)$, but does not intersect S. Let x be the point where $\hat{\beta}$ first enters $Nb_{\hat{X}}(K, S)$, and let y be the point where $\hat{\beta}$ last exits $Nb_{\hat{X}}(S, K)$. Denote by $\hat{\beta}_{xy}$ the segment of $\hat{\beta}$ joining x and y. Then $l_{\hat{X}}(\hat{\beta}_{xy}) \leq \hat{d}(x, y) \leq 2K$. Since $R > 3K$, the geodesic $\hat{\beta}_{xy}$ does not leave $Nb_{\hat{X}}(S, R)$, so that $\hat{\beta}_{xy}$ does not intersect any subset $S' \in \Sigma$, and by Lemma 2.7, $l_S(Pr(\beta_{xy}, S)) \leq D(1)$. Observe that $\hat{\beta}$ is the concatenation of $\hat{\beta}_{xy}$ and other two (possibly, degenerate) segments that do not intersect $Nb_{\hat{X}}(S, K)$. Therefore, we have that $l_S(Pr(\hat{\beta}, S)) \leq D(1) + 2E_1$ in this case.

Finally, assume that $\hat{\beta}$ intersects S. Let x_s (or y_s) be the point where $\hat{\beta}$ first enters (or last exits) S. As $\hat{X}$ is a hyperbolic space, $\hat{\alpha}$ and $\hat{\beta}$ stay a bounded distance apart; let $M(P)$ be a constant that bounds this distance. There are points $z, w \in \hat{\alpha}$ with $\hat{d}(z, x_s) \leq M(P)$ and $\hat{d}(w, y_s) \leq M(P)$. Let $\hat{\alpha}_{zw}$ be the segment of $\hat{\alpha}$ between z and w, we have that $l_{\hat{X}}(\hat{\alpha}_{zw}) \leq 2PM(P)$. By Lemma 2.8,

$$l_S(Pr(\alpha_{zw}), S) \leq E(\hat{\alpha}_{zw}) \leq E(2PM(P)).$$

Let $\hat{\alpha}_1$ (or $\hat{\alpha}_2$) be the segment of $\hat{\alpha}$ joining the initial point of $\hat{\alpha}$ and z (or w and the terminal point of $\hat{\alpha}$). Let β_i be an X-geodesic joining the endpoints of $\hat{\alpha}_i$. W.l.o.g. (cf. the proof of Lemma 2.10), we can assume that β_1, β_2 do not intersect S. As we have shown, $l_S(Pr(\beta_i, S)) \leq D(1) + 2E_1$, for $i = 1, 2$.

Therefore, $l_S(Pr(\alpha, S)) \leq 2(D(1) + 2E_1) + E(2PM(P))$, so that

$$s \leq 3(2(D(1) + 2E_1) + E(2PM(P))) \tag{9}$$

which finishes the proof in the case (1).

Now, we prove (2). Let s be the X-distance between the points where ξ and τ first enter the subset S, and let ξ_1 and τ_1 be the initial segments of ξ and τ with ends at the points where these paths first enter S. Observe that $s \leq l_S(Pr(\xi_1^{-1} \circ \tau_1), S)$. The arguments used in the proof of case (1) show that

$$s \leq 2(2(D(1) + 2E_1) + E(2PM(P))).$$

Obviously, the distance between the points where ξ and τ last exit the subset S is bounded by $2(2(D(1) + 2E_1) + E(2PM(P)))$ as well.

Therefore, one can set $B = 3(2(D(1) + 2E_1) + E(2PM(P)))$ to finish the proof of the lemma. □

3. Proof of Theorem 1.6

First, consider the case of one subgroup H so that G is hyperbolic relative to H. By [**6**, Corollary 3.2], we can assume that the generating set A of G contains a generating set A_H of H as a subset. We define a map f from the Cayley graph $\Gamma(G, A)$ to X as follows. Lift the ray in the quotient space Y to a

ray $\gamma\colon [0,\infty) \to X$ and choose a horoball S so that the boundary of S is the level surface of the horofunction which corresponds to the ray γ, and the images GS of S under the action of G form a G-invariant system of disjoint horoballs. We assume that the minimum distance between two horoballs is R. Notifying that a horoball is a convex subset of X, we define X' to be the space obtained from X by deleting the interiors of all the horoballs in $\Sigma = GS$ (cf. notation 2.1). Clearly, the action of G on X' is cocompact.

Pick a point x on the boundary of S. Let $g.x$ be the image of x under the action of $g \in A$. Hence, $g.x \in g.S$. Observe that the horoball $g.S$ corresponds to the left coset gH in the following meaning. If $g \in A \setminus A_H$, then $g.S \neq S$, and we join x and $g.x$ by a segment which does not intersect any horoball. If $g \in A_H$, then $g.S = S$, and we join x and $g.x$ by a segment which lies in S. Define a map $f\colon \Gamma(G,A) \to X$ as follows. Let $id \in \Gamma(G,A)$ be the point that corresponds to 1_G. The map f sends $id \in \Gamma(G,A)$ to the point $x \in X$ and $[id, g.id]$ to the segment in X that joins x and $g.x$, for each $g \in A$. Extend f to $\Gamma(G,A)$ equivariantly. Since G acts on X by isometries, and the action of G on $\hat{X}$ is cocompact, the map f induces a quasi-isometry $\hat{f}$ between $\hat{\Gamma}$ and $\hat{X}$. Therefore, [**9**, Proposition 1] (cf. Lemma 2.2) implies that $\hat{\Gamma}$ is a hyperbolic metric space i.e., G is weakly hyperbolic relative to H ([**9**, Theorem 1], cf. Theorem 1.2).

Furthermore, let ρ be a path in $\Gamma(G,A)$, and let $f(\rho) = \beta \in X$. If ρ penetrates a coset gH, then β intersects the horosphere $g.S$; moreover, if the distance between the points where β first enters and last exits the boundary of $g.S$ is bounded by l, then the Γ-distance that ρ travels in gH is bounded in terms of l and the constants of quasi-isometry $\hat{f}$. Therefore, the pair (G,H) has the BCP property.

Now, assume that G is hyperbolic relative to $H_1,\dots,H_k$, in the sense of the Gromov definition. We assume that the generating set A of G contains a generating set $A_i \subset A$ for H_i, for all $i = 1,\dots,k$. Lift the k rays in the quotient space Y to rays $\gamma_i\colon [0,\infty) \to X$ and choose horoballs S_i so that the boundary of S_i is the level surface of the horofunction which corresponds to the ray γ_i, and the images $\Sigma = \bigcup_{i=1}^k GS_i$ of S_i under the action of G form a G-invariant system of disjoint horoballs. Assume that the distance between two horoballs in Σ is bounded below by R. Pick a point x_i in S_i, for each $i = 1,\dots,k$, and connect each x_j to x_1 by an edge e_j, for $j = 2,\dots,k$. The union of the edges e_j is a finite tree which we denote by T. Each generator $h \in A_i$ (for some i) maps $x_i \in S_i$ to another point $y_i \in S_i$, and $x_j \in S_j$ $(j \neq i)$ to a point $h.x_i \in h.S_i \neq S_i$, so that hT and T are disjoint. Obviously, T and $g.T$ are disjoint for each generator $g \in A \setminus \bigcup_{i=1}^k A_i$. We proceed as in the case of one subgroup and get a map from $\hat{\Gamma}$ to $\hat{X}$. Contract the tree T and all its images GT to a point, in order to see that $\hat{f}$ is a quasi-isometry between $\hat{\Gamma}$ and $\hat{X}$. The same argument as above shows that the BCP property holds.

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